The King's Diagonal

Three Parents Instead of Two

How the Chess King Escalates Pascal's Triangle to that of Delannoy

Lee McCulloch-James
Department of Natural & Unnatural Sciences
Dwight School London

May 24, 2026

Abstract

We pose a simple chessboard question: in how many ways can a king cross the board diagonally, using only forward moves? From this we develop the full combinatorial machinery of the Delannoy triangle, in patient parallel to Pascal's. The exposition is aimed at sixth-form and early undergraduate students. We construct the triangle from a three-parent recurrence, prove a closed binomial formula, and verify it against the chessboard count of 48 639. We then turn to the central spines of both triangles: the central binomial coefficients $\binom{2n}{n}$, which grow like 4^n, and the central Delannoy numbers $D(n,n)$, which grow like $(3+2\sqrt{2})^n$. The Pascal rate $4 = 2^2$ is rational; the Delannoy rate $3+2\sqrt{2} = (1+\sqrt{2})^2$ is an algebraic irrational, the square of the silver mean. We show through the transfer matrices of the two recurrences that this is a structural escalation: admitting one extra local dependency in the recurrence promotes a scalar dynamic into a genuinely spectral one, with the silver mean appearing as an eigenvalue. No knowledge beyond binomial coefficients is assumed.

1 The Question

Place a king in the corner a1 of an empty chessboard. He is to walk to the diagonally opposite corner h8. To keep things finite and well-defined, we restrict his legal moves to those that make progress toward the goal:

- one square East, one square North, one square North-East (the diagonal).

He may never retreat. *How many distinct walks are available to him?*

The answer though you might suspect small at 48 639, is large enough to surprise but small enough to verify by a hand-built table. Our task is to understand *why* it is that number, and to see the structure that makes the answer compressible into a few lines of arithmetic.

2 The Two-Step Cousin: Pascal's Triangle

Before introducing the diagonal, consider the simpler problem where the king moves only East or North, effectively a rook walking forward.

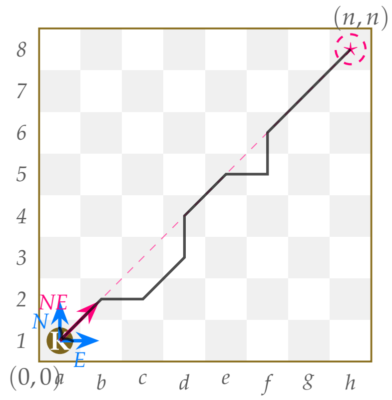


Figure 1: The king walks from a1 to h8 using only East, North, and North-East moves. Three legal first moves are shown in colour; one example completed path is drawn faintly across the board. The dashed line marks the Euclidean geodesic.

A rook-walk from $(0,0)$ to (n,k) consists of n North-steps and k East-steps in some order. The number of such walks is the number of ways to choose which k of the $n+k$ moves are East:

$$R(n,k) = \binom{n+k}{k}. \tag{1}$$

For the chessboard corner-to-corner traversal $(7,7)$ this gives $\binom{14}{7} = 3432$.

The values $R(n,k)$ are the entries of Pascal's triangle, traditionally displayed as a centred isoceles arrangement so the symmetry is visible:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Each entry is the sum of the two immediately above it, encoding the recurrence

$$R(n,k) = R(n-1,k) + R(n,k-1), \tag{2}$$

which says the last step of any walk to (n,k) was either a North-step (arriving from $(n-1,k)$) or an East-step (arriving from $(n,k-1)$). Sum the two predecessor counts; you have the current count.

The central binomial spine

Running down the centre of Pascal's triangle is the sequence

$$\binom{2n}{n} = 1,\ 2,\ 6,\ 20,\ 70,\ 252,\ 924,\ 3432,\ 12870,\ \ldots \tag{3}$$

These are the **central binomial coefficients**. Geometrically they count the rook-walks from $(0,0)$ to (n,n) on an $n \times n$ square, because such a walk must interleave exactly n East-steps among n North-steps. The chessboard answer $\binom{14}{7} = 3432$ is the eighth term of this spine, sitting at row $n = 7$. Three properties of this spine matter for what follows.

Closed form. The exact recurrence

$$\binom{2n}{n} = \frac{2(2n-1)}{n}\binom{2(n-1)}{n-1} \tag{4}$$

expresses each entry in terms of its predecessor, which allows the consecutive ratios to be computed directly.

Consecutive ratios. The ratio of one term to the next is

$$\frac{\binom{2n}{n}}{\binom{2(n-1)}{n-1}} = 4 - \frac{2}{n}, \tag{5}$$

which crawls toward 4 from below. Numerically:

n	$\binom{2n}{n}$	ratio	gap to 4	ratio / 4
1	2	2.000	2.000	0.500
2	6	3.000	1.000	0.750
3	20	3.333	0.667	0.833
4	70	3.500	0.500	0.875
5	252	3.600	0.400	0.900
6	924	3.667	0.333	0.917
7	3432	3.714	0.286	0.929

The ratios are settling toward the limit, but the gap to 4 closes only at the rate $2/n$.

Asymptotic. The Stirling approximation gives the leading-order behaviour

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}} \tag{6}$$

which says the spine grows like 4^n divided by a slow $\sqrt{n}$ correction. The 4 is the dominant rate; the $1/\sqrt{n}$ is the reason consecutive ratios approach 4 rather than equal it.

The number 4 deserves a moment of attention. Each rook-step is one of two choices, and reaching (n, n) takes $2n$ steps, so the total number of length-$2n$ walks the rook *could* make (without the constraint of ending at (n, n)) is $2^{2n} = 4^n$. The central binomial coefficient is the fraction of those walks that hit the target. The asymptotic $4^n/\sqrt{\pi n}$ records this exactly: the spine grows like the total walk count, attenuated by the $1/\sqrt{n}$ probability of landing on the diagonal.

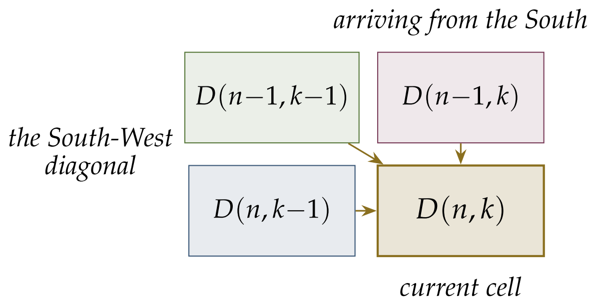
So Pascal's spine grows like $2^2 = 4$ because the rook has 2 choices per step and the spine entry sits two indices' worth of growth from the origin.

3 The Three-Parent Rule

Now restore the diagonal step. A walk to (n, k) may terminate in any of *three* final moves:

- East from $(n, k-1)$; North from $(n-1, k)$; North-East from $(n-1, k-1)$.

Each cell is fed by exactly three neighbours and to determine $D(n, k)$ one needs only the three immediately adjacent precomputed cells. The rule is local in that no global information is required, no backtracking. This is the same property that makes Pascal's triangle computable



by elementary-school arithmetic, with the addition of one extra parent. Each predecessor contributes its own count of walks to define the **Delannoy numbers**, [1], [2], [3], [5] as:

$$\boxed{D(n,k) = D(n,k-1) + D(n-1,k) + D(n-1,k-1)} \tag{7}$$

with the boundary condition $D(0,0) = 1$ and $D(n,k) = 0$ whenever n or k is negative.

Along the axes, only one walk exists, straight East or straight North, so $D(n,0) = D(0,k) = 1$. The interior is then determined entirely by (7). Filling cells row by row from the origin gives the following grid:

$n\backslash k$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	3	5	7	9	11	13	15
2	1	5	13	25	41	61	85	113
3	1	7	25	63	129	231	377	575
4	1	9	41	129	321	681	1289	2241
5	1	11	61	231	681	1683	3653	7183
6	1	13	85	377	1289	3653	8989	19825
7	1	15	113	575	2241	7183	19825	**48639**

The entry $D(7,7) = 48\,639$ in the bottom-right corner is our chessboard answer. We can verify a few cells by hand to confirm the rule.

- $D(1,1) = D(1,0) + D(0,1) + D(0,0) = 1 + 1 + 1 = 3$. Walks to (1,1): East-then-North, North-then-East, or one diagonal.
- $D(2,2) = D(2,1) + D(1,2) + D(1,1) = 5 + 5 + 3 = 13$.
- $D(7,7) = D(7,6) + D(6,7) + D(6,6) = 19825 + 19825 + 8989 = 48\,639$.

4 The Central Delannoy Spine

Following the diagonal $n = k$ down the table yields the **central Delannoy numbers**:

$$D(n,n) = 1,\ 3,\ 13,\ 63,\ 321,\ 1683,\ 8989,\ 48639,\ 265729,\ \ldots \tag{8}$$

This is the Delannoy analogue of Pascal's central spine. Each term counts the king-walks across an $n \times n$ board. The chessboard answer $D(7,7) = 48{,}639$ sits at row $n = 7$, directly comparable to the rook's $\binom{14}{7} = 3432$.

Noting a few features of the spine will repay our attention.

Divisibility. Every term after the first appears to be divisible by 3:

$$3,\ 13,\ 63 = 3 \cdot 21,\ 321 = 3 \cdot 107,\ 1683 = 3 \cdot 561,\ 8989 = ?$$

But at $n = 6$ the divisibility is more subtle: $8989 = 89 \cdot 101$, so the divisibility by 3 already fails. The stronger statement is that $D(n,n) \equiv 1 \pmod 3$ when n is divisible by 3, and a small modular calculation handles the remaining residues. We mention this only as a flag that the spine carries genuine arithmetic content; we will not pursue the proof here.

Consecutive ratios. As with Pascal's spine, the ratio of consecutive terms is the natural measure of growth. Computing:

n	$D(n,n)$	ratio	gap to $3+2\sqrt{2}$	ratio / $(3+2\sqrt{2})$
1	3	3.000	2.828	0.515
2	13	4.333	1.495	0.743
3	63	4.846	0.982	0.831
4	321	5.095	0.733	0.874
5	1683	5.243	0.585	0.900
6	8989	5.341	0.487	0.916
7	48639	5.411	0.417	0.928
8	265729	5.463	0.365	0.937

The shape of this convergence is uncannily similar to Pascal's. Both spines approach their asymptotic ratio from below, both close their gap at a rate roughly proportional to $1/n$, and both pass through ratio-over-limit values of 0.93 at $n = 7$. The asymptotic ratio itself differs, but the convergence dynamics do not as both sequences are diagonal extractions of a bivariate generating function, and both have asymptotics of the form

$$a_n \sim \frac{\rho^n}{C\sqrt{n}} \tag{9}$$

where ρ is the dominant growth rate and C is a constant. The $1/\sqrt{n}$ correction is what makes both ratios drag.

Asymptotic. The saddle-point analysis of the bivariate Delannoy generating function (which we treat in the next section) yields

$$D(n,n) \sim \frac{(3+2\sqrt{2})^n}{\sqrt{\pi n\,(3\sqrt{2}-4)}} \tag{10}$$

with $3+2\sqrt{2} = (1+\sqrt{2})^2$ as the dominant rate. The denominator constant $\sqrt{\pi n(3\sqrt{2}-4)}$ matches the structural role of the $\sqrt{\pi n}$ in (6).

Side by side. The parallel between the two spines looks thus:

	Spine	Asymptotic growth
Pascal (rook, two choices)	$\binom{2n}{n}$	$4^n/\sqrt{\pi n}$
Delannoy (king, three choices)	$D(n,n)$	$(3+2\sqrt{2})^n/\sqrt{\pi n(3\sqrt{2}-4)}$

The growth rates 4 and $3+2\sqrt{2}$ are both *squared expansion factors*: the rook's is $2^2 = 4$, with the 2 coming from its two-choice step, while the king's is $(1+\sqrt{2})^2 = 3+2\sqrt{2}$, with the $1+\sqrt{2}$ being the silver mean. The squaring is the same in both cases, because to reach the spine entry at row n one must do n worth of expansion on each of the two axes. What changes is the per-step factor: a rational 2 for the rook, the algebraic silver mean due to the diagonal step for the king. We will see where the silver mean comes from in section 6.

5 A Closed Formula

The recurrence is enough to compute any single value, but it would be more satisfying to have a closed expression. The key observation to achieve this is that any walk to (n, k) can be classified by the number of diagonal steps it contains. Suppose a walk uses exactly j diagonal moves. Each diagonal contributes one unit to both coordinates, so the remaining moves must contribute $k-j$ East-units and $n-j$ North-units. The total number of moves is therefore $j+(n-j)+(k-j) = n+k-j$.

To count walks with exactly j diagonals: choose which j of the $n+k-j$ slots are diagonals $\binom{n+k-j}{j}$, then choose which of the remaining $n+k-2j$ slots are East-steps $\binom{n+k-2j}{k-j}$. Multiplying and summing over all valid j:

Proposition 1. *For $n, k \geq 0$,*

$$D(n,k) = \sum_{j=0}^{\min(n,k)} \binom{n+k-j}{j}\binom{n+k-2j}{k-j}. \tag{11}$$

Remark. An equivalent and often more convenient form is

$$D(n,k) = \sum_{j=0}^{\min(n,k)} \binom{n}{j}\binom{k}{j} 2^j,$$

obtained by reorganising the choice as "pick the j diagonals from the n North-units and the k East-units, each diagonal merging one of each." The factor 2^j tracks the freedom to interleave.

Verification at the chessboard

Apply the formula with $n = k = 7$:

$$D(7,7) = \sum_{j=0}^{7} \binom{7}{j}^2 2^j.$$

The seven terms are tabulated below showing closed formula and recurrence agree.

j	$\binom{7}{j}^2$	2^j	contribution
0	1	1	1
1	49	2	98
2	441	4	1764
3	1225	8	9800
4	1225	16	19600
5	441	32	14112
6	49	64	3136
7	1	128	128
		total	**48639**

6 Where the Silver Mean Comes From

The Pascal spine grows like 4 and the Delannoy spine grows like $3 + 2\sqrt{2}$. Both are squared expansion factors, but the per-step expansion changes from the integer 2 to the silver mean $1 + \sqrt{2}$. We now show why.

The cleanest derivation arrives by way of *generating functions* which the appendix elaborates. For Pascal, the two-parent recurrence is encoded by

$$\sum_{n,k\geq 0} R(n,k)\, x^n y^k \;=\; \frac{1}{1-x-y}. \tag{12}$$

The denominator records the two allowed step-monomials, x for North and y for East. Restricting to the diagonal $n = k$ amounts to setting $x = y = z$, which gives $1/(1-2z)$, a geometric series with dominant pole at $z = 1/2$. The radius of convergence is therefore $1/2$, and the diagonal coefficients grow like $(1/(1/2))^{2n} = 4^n$. The 4 is the *square* of the inverse radius because each z carries one unit of both indices, so the rate per unit of n is doubled.

For Delannoy, the three-parent recurrence is encoded by

$$\sum_{n,k\geq 0} D(n,k)\, x^n y^k \;=\; \frac{1}{1-x-y-xy}. \tag{13}$$

The new xy monomial records the diagonal step, which contributes one unit to each index simultaneously. Setting $x = y = z$ gives

$$\frac{1}{1-2z-z^2} \tag{14}$$

whose denominator is a *quadratic* in z, not linear as in the Pascal case. The dominant pole lies at the positive root of $z^2 + 2z - 1 = 0$, namely

$$z_\star \;=\; -1+\sqrt{2} \;=\; \frac{1}{1+\sqrt{2}}, \tag{15}$$

the reciprocal of the silver mean. The radius of convergence is therefore $\sqrt{2} - 1$, the inverse-radius is $1 + \sqrt{2}$, and squaring gives the spine growth rate

$$(1+\sqrt{2})^2 \;=\; 3+2\sqrt{2} \;\approx\; 5.828. \tag{16}$$

The silver mean appears precisely because, within the forward-progress framework we have imposed, the third (diagonal) step promotes the characteristic equation from linear to quadratic. The framework matters. If the king were allowed his full eight moves on an unconstrained board, the problem would cease to be a Delannoy-type lattice-path enumeration and become a random walk, where paths may revisit cells arbitrarily and no comparable spine structure survives.

Within the forward lattice, however, the silver mean is genuinely forced. Even replacing the symmetric $(1,1)$ diagonal by an asymmetric step such as $(1,2)$ would change the characteristic equation and produce a different algebraic growth rate.

The Delannoy king is a directed (teleologically directed) king. The admissible moves impose a partial ordering on the lattice, forbidding cycles and thereby converting the walk into a recursively enumerable structure.

The growth rate is controlled by the nearest point where the generating function ceases to behave analytically. In rational generating functions this occurs when the denominator vanishes.

A heuristic reading of the edge-sum

Before turning to the algebra, it is worth pausing to ask whether geometric intuition could have predicted the expansion factor. A natural first guess is that the per-step expansion should reflect the total "available movement" the king has at each cell. Pascal's lattice offers two unit directions, North and East, giving the heuristic sum $1+1 \;=\; 2$, which is exactly the Pascal expansion factor. The Delannoy lattice adds a diagonal of Euclidean length $\sqrt{2}$, suggesting the heuristic quantity

$$1+1+\sqrt{2} \;=\; 2+\sqrt{2} \;\approx\; 3.414.$$

But the actual expansion factor is

$$1+\sqrt{2} \;\approx\; 2.414,$$

not the heuristic sum. The geometric reading is suggestive but not exact.The true growth rate is determined not by geometric edge lengths directly, but by the dominant eigenvalue of the transfer operator encoded by the recurrence. So while the geometry hints, the algebra decides.

Where Pascal and Delannoy diverge: scalar versus matrix transfer

The real conceptual centre is not the numerical coincidence between $\sqrt{2}$ in the lattice metric and $\sqrt{2}$ in the algebra. It is the following structural fact:

Adding one extra local dependency to the recurrence promotes a scalar recursion into spectral dynamics.

This is what changes when the diagonal step is admitted. We see it most clearly by writing each spine recurrence as a *transfer system* and observing what the transfer operator does.

Pascal: first-order growth, scalar transfer. The Pascal spine satisfies, to leading order, the simple multiplicative recurrence

$$\binom{2n}{n} \;\approx\; 4\binom{2(n-1)}{n-1}.$$

This is a first-order recurrence: each term depends only on the one before it. The transfer object is therefore a 1×1 scalar

$$T_{\text{Pascal}} \;=\; \big[\,4\,\big].$$

A scalar has just one eigenvalue, equal to itself. There is no spectral structure to speak of: trace, determinant, dominant eigenvalue, and the scalar itself are all the same number, 4.

Delannoy: second-order growth, matrix transfer. The Delannoy spine satisfies, to leading order, the second-order recurrence

$$D(n,n) \;\approx\; 6\,D(n-1,n-1) \;-\; D(n-2,n-2).$$

Two previous terms now feed each new one. To write this as a transfer system we must lift to a state vector of size two:

$$\begin{pmatrix} D(n+1,n+1) \\ D(n,n) \end{pmatrix} \;=\; T_{\text{Delannoy}} \begin{pmatrix} D(n,n) \\ D(n-1,n-1) \end{pmatrix}, \qquad T_{\text{Delannoy}} \;=\; \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}.$$

The top row encodes the recurrence; the bottom row carries the previous top entry down for use next step. The transfer operator is now a genuine 2×2 matrix.

The eigenvalue is now structurally unavoidable. For the scalar T_{Pascal} there was nothing to diagonalise. For T_{Delannoy} the growth rate is encoded in the spectrum. The characteristic polynomial

$$\det(T_{\text{Delannoy}} - \lambda I) \;=\; \lambda^2 - 6\lambda + 1$$

has roots

$$\lambda_\pm \;=\; 3 \pm 2\sqrt{2} \;=\; (1 \pm \sqrt{2})^2.$$

The dominant eigenvalue $\lambda_+ = 3 + 2\sqrt{2}$ is the spine growth rate. The sub-dominant eigenvalue $\lambda_- = 3 - 2\sqrt{2} \approx 0.172$ is a decaying mode that explains the slow approach of consecutive ratios from below: the spine inherits both contributions, but only the dominant one survives in the limit.

The contrast in one table. The full pedagogical arc is visible at a glance:

Structure	Pascal	Delannoy
Parents in recurrence	2	3
Recurrence order on the spine	1	2
Transfer object	scalar [4]	matrix $\begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$
Dominant growth rate	rational integer 4	quadratic irrational $3 + 2\sqrt{2}$
Spectral structure	trivial	non-trivial
Lattice cousin	rook	king

In the scalar case, trace and eigenvalue coincide trivially: nothing forces them apart. In the matrix case they separate. The trace of T_{Delannoy} is 6, the sum of both eigenvalues, while the dominant eigenvalue alone is $3 + 2\sqrt{2} \approx 5.828$. The heuristic edge-sum reading of the previous subsection is, in retrospect, attempting to compute something trace-like by adding up geometric edge lengths, but the lattice's spine growth listens to the algebra of the characteristic equation, not to any direct geometric summation.

So we have that the silver mean is real, but it is not where the action is. The action is in the promotion of a scalar recursion to a matrix one, with the eigenvalue emerging not as an embellishment but as a structurally unavoidable consequence of admitting one extra local dependency in the recurrence.

The Pell-recurrence skeleton

The silver mean satisfies $\alpha^2 = 2\alpha + 1$, the defining equation of the silver-mean family. This is the algebraic shadow of the recurrence

$$a_n \;=\; 6a_{n-1} - a_{n-2}, \tag{17}$$

whose characteristic polynomial $\lambda^2 - 6\lambda + 1$ has roots $3 \pm 2\sqrt{2} \;=\; (1 \pm \sqrt{2})^2$. The central Delannoy numbers do not quite satisfy this recurrence exactly (the true Delannoy recurrence has more structure), but they share its dominant eigenvalue. The same recurrence governs the Pell numbers $1, 2, 5, 12, 29, 70, \ldots$, the silver-mean continued-fraction convergents, and the balancing numbers of Behera and Panda. The Delannoy spine sits inside this *Pell family* the way the central binomial coefficients sit inside the Pascal family. The diagonal step is what promotes the structure from Pascal to Pell, from gold to silver, from a rational growth rate to an algebraic one.

7 Why So Much Larger Than Pascal?

The rook-only walk count is $\binom{14}{7} = 3432$. The king's count is 48 639. The ratio is

$$\frac{48\,639}{3432} \approx 14.2.$$

The diagonal step compounds as every interior cell has three predecessors instead of two, and the counts feed forward multiplicatively. By row $n = 20$ the king's count exceeds the rook's by a factor of order 10^7. By $n = 50$ the gap is astronomical.

This compounding is exactly the difference between the growth rates 4 and 5.83 raised to large powers. For large n,

$$\frac{D(n,n)}{\binom{2n}{n}} \sim \left(\frac{3+2\sqrt{2}}{4}\right)^n \cdot C \approx 1.457^n \cdot C \tag{18}$$

where C is a constant of order one absorbing the $\sqrt{n}$ corrections. Each step compounds the king's lead over the rook by about 46%.

8 Closing Thought

The chessboard problem has the cleanest possible answer: *three parents, one rule, build the table.* Every cell is just the local sum of its three predecessors, and the global count emerges by recursion. The same logic gives $D(2,2) = 13$ by hand and $D(7,7) = 48\,639$ by table, with no algebraic trickery.

What makes the Delannoy numbers worth attention beyond this exercise is the arithmetic that grows on top of the geometry: the central values $1, 3, 13, 63, 321, 1683, \ldots$ also count king-walks across a chessboard, appear as evaluations of Legendre polynomials at the value 3, satisfy non-trivial congruences modulo primes, and share their bivariate generating-function structure with the Schröder numbers from one-dimensional walk theory.

But all of that begins with the same observation we made in section 7: three parents, one rule. The rest is just following the consequences.

Exercise. A bishop moving only on diagonals can never leave the colour of its starting square. Construct a forward-directed bishop walk using only North-East and South-East moves. What recurrence governs the resulting lattice? How does the parity colouring split the board into disconnected components?

Exercise. Estimate $D(10,10)$ first using the asymptotic

$$D(n,n) \sim \frac{(3+2\sqrt{2})^n}{\sqrt{\pi n(3\sqrt{2}-4)}},$$

then compute the exact value by both the recurrence table and Proposition 1. Compare the error in the asymptotic estimate. Hint: Compute $D(10,10)$ in two ways: by extending the table, and by Proposition 1. Confirm both give 8 097 453.

Exercise. Show directly from the closed formula that $D(n,k) = D(k,n)$. What property of king-walks does this symmetry reflect?

Exercise. Verify by hand that the consecutive-ratio formula $\binom{2n}{n}/\binom{2(n-1)}{n-1} = 4 - 2/n$ holds at $n = 5$. Then compute the corresponding ratio $D(5,5)/D(4,4)$ and observe that it does not simplify to a clean rational of the form $a - b/n$, despite approaching the same kind of limit. What does this tell you about the difference between the two spines?

Exercise. Replace the symmetric diagonal step $(1,1)$ by the directed skew step $(1,2)$. Construct the corresponding recurrence relation and compute the first few central values. Does the dominant growth still appear to be quadratic irrational, or does a higher algebraic degree emerge?

Exercise. Starting from the recurrence

$$D(n,k) = D(n-1,k) + D(n,k-1) + D(n-1,k-1),$$

derive the generating function

$$\sum_{n,k\geq 0} D(n,k)x^n y^k = \frac{1}{1-x-y-xy}.$$

Explain explicitly why the term xy appears.

Exercise. Pascal's triangle has two parent cells. Delannoy's triangle has three. Design a four-parent directed lattice recurrence of your own. What generating function does it produce? What kind of asymptotic growth appears along its central spine?

Remark. Pascal and Delannoy inherit equally from their parent cells. Other triangles, such as the Worpitzky triangle, introduce position-dependent weighting into the inheritance rule itself. One may ask whether a "weighted Delannoy" analogue exists, and what spectral structures its transfer operators would possess.

References

[1] H. Delannoy, "Sur le nombre de chemins du cavalier," *Association Française pour l'Avancement des Sciences*, vol. 24, pp. 41–42, 1895.

[2] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, 1974. See Chapter V for lattice paths and Delannoy numbers.

[3] R. P. Stanley, *Enumerative Combinatorics, Volume 1*, 2nd ed., Cambridge University Press, 2011. Sections on lattice-path enumeration and generating functions.

[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009. Chapters III–VI on generating functions, singularities, and asymptotic growth.

[5] N. J. A. Sloane et al., *OEIS Sequence A001850: Central Delannoy Numbers*, `https://oeis.org/A001850`

Appendix: Generating Functions as Encodings of Recurrences

A useful way to think about generating functions is this:

> *Generating functions are algebraic encodings of shift recurrences. A recurrence relation becomes a polynomial denominator once shifts are translated into multiplication by variables.*

The connection between dominant singularities and coefficient growth is a standard result of analytic combinatorics [4]. At first, expressions such as

$$\frac{1}{1-x-y} \qquad \text{or} \qquad \frac{1}{1-x-y-xy}$$

can appear to arrive mysteriously, as though deigned to be true handed down. In this appendix we show how their form is a direct algebraic shadow of the lattice recurrence itself.

Pascal's triangle

Pascal's recurrence is

$$R(n,k) = R(n-1,k) + R(n,k-1).$$

Define the bivariate generating function

$$G(x,y) = \sum_{n,k\geq 0} R(n,k)x^n y^k.$$

The variables x and y are bookkeeping devices:

$$x \;\leftrightarrow\; \text{one step in the } n\text{-direction,} \qquad y \;\leftrightarrow\; \text{one step in the } k\text{-direction.}$$

Expanding the first few terms gives

$$G(x,y) = 1 + x + y + 2xy + x^2 + y^2 + 3x^2y + 3xy^2 + \cdots$$

Now multiply the whole series by x:

$$xG(x,y) = x + x^2 + xy + 2x^2y + x^3 + xy^2 + \cdots$$

Every coefficient has shifted one power to the right in x. The coefficient that previously occupied lattice position (n,k) now occupies $(n+1,k)$. Multiplication by x therefore implements a one-step shift in the n-direction. For example, the Pascal coefficient $R(1,1) = 2$ appears as $2xy$. After multiplication by x, it becomes

$$2x^2y,$$

which now sits at lattice position $(2,1)$. Similarly:

$$yG(x,y)$$

shifts coefficients one step in the k-direction. The recurrence therefore becomes

$$G(x,y) = xG(x,y) + yG(x,y) + 1.$$

The extra 1 records the boundary condition $R(0,0) = 1$. Rearranging:

$$(1-x-y)G(x,y) = 1,$$

hence

$$G(x,y) = \frac{1}{1-x-y}.$$

The denominator is thus simply the recurrence relation written algebraically.

The Delannoy extension

For the Delannoy lattice the recurrence gains one additional predecessor:

$$D(n,k) = D(n-1,k) + D(n,k-1) + D(n-1,k-1).$$

Repeating the same procedure gives

$$G(x,y) = xG(x,y) + yG(x,y) + xyG(x,y) + 1.$$

The new term $xyG(x,y)$ appears because the diagonal step shifts both indices simultaneously. Thus:

$$(1-x-y-xy)G(x,y) = 1,$$

giving

$$G(x,y) = \frac{1}{1-x-y-xy}.$$

Each monomial in the denominator corresponds directly to an admissible predecessor move:

$$x \leftrightarrow (1,0), \qquad y \leftrightarrow (0,1), \qquad xy \leftrightarrow (1,1).$$

Why singularities determine growth

The generating function now contains the asymptotic growth information of the lattice. The key idea is that the coefficients of a power series are controlled by the nearest point where the function ceases to behave analytically: that being the nearest singularity. For Pascal:

$$G(z,z) = \frac{1}{1-2z}.$$

The denominator vanishes at $z = \frac{1}{2}$. This singularity controls the long-run growth. Its reciprocal gives the per-axis expansion factor: $\frac{1}{1/2} = 2$. Since the central spine advances simultaneously in both indices, the total growth becomes

$$2^2 = 4.$$

For Delannoy:

$$G(z,z) = \frac{1}{1-2z-z^2}.$$

The denominator vanishes when

$$z^2 + 2z - 1 = 0,$$

whose positive root is

$$z_\star = \sqrt{2} - 1 = \frac{1}{1+\sqrt{2}}.$$

The reciprocal therefore gives the dominant per-axis expansion factor: $1+\sqrt{2}$, and squaring yields the Delannoy spine growth rate

$$(1+\sqrt{2})^2 = 3 + 2\sqrt{2}.$$

The silver mean therefore emerges because the Delannoy recurrence shifts the nearest singularity of the generating function.

Transfer matrices and recurrence order

The same structure appears spectrally. Pascal's central spine behaves asymptotically like a first-order recurrence:

$$a_n \approx 4a_{n-1}.$$

Its transfer operator is therefore the trivial 1×1 matrix

$$T_{\text{Pascal}} = (4).$$

The Delannoy spine instead behaves like a second-order recurrence:

$$a_n \approx 6a_{n-1} - a_{n-2}.$$

Second-order recurrences require a two-component state vector:

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}.$$

The characteristic polynomial becomes

$$\lambda^2 - 6\lambda + 1 = 0,$$

with roots

$$3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2.$$

The dominant eigenvalue reproduces the same asymptotic growth obtained from the generating-function singularity. The generating-function and transfer-matrix viewpoints are therefore two expressions of the same underlying spectral structure.

A final structural remark

The silver mean is a fingerprint of the particular forward-directed lattice structure we have imposed. Our king is teleologically constrained: every admissible move advances toward a prescribed endpoint. The resulting lattice is directed and acyclic, which is precisely what allows recurrences, generating functions, and transfer matrices to govern the counting problem. The silver mean appears specifically because one symmetric diagonal move couples the two forward lattice directions into a quadratic recurrence. It is the algebraic shadow of that coupling.